

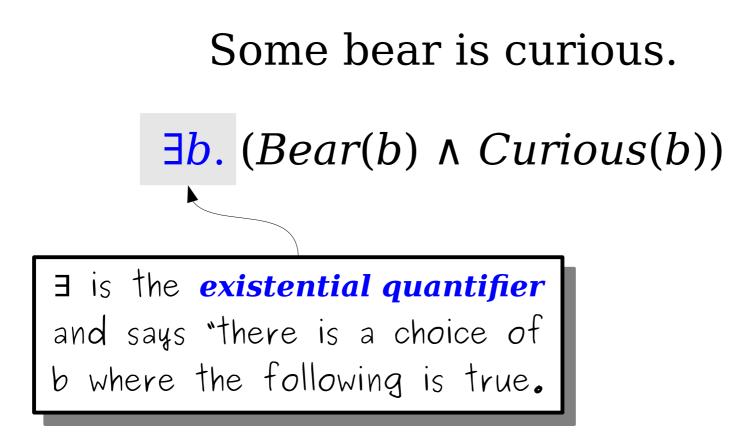
Lecture 05: First-Order Logic

Part 2 of 2

Recap from Last Time

What is First-Order Logic?

- **First-order logic** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
 - predicates that describe properties of objects,
 - *functions* that map objects to one another, and
 - *quantifiers* that allow us to reason about many objects at once.



"For any natural number n, n is even if and only if n^2 is even"

 $\forall n. \ (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$

∀ is the universal quantifier and says "for any choice of n, the following is true."

"Some P is a Q"

translates as

 $\exists x. (P(x) \land Q(x))$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$\exists x. (P(x) \land Q(x))$

If x is an example, it *must* have property P on top of property Q.

"All P's are Q's"

translates as

 $\forall x. \ (P(x) \rightarrow Q(x))$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$\forall x. \ (P(x) \rightarrow Q(x))$

If x is a counterexample, it must have property P but not have property Q.

New Stuff!

The Aristotelian Forms

"All As are Bs" $\forall x. (A(x) \rightarrow B(x))$ "Some As are Bs"
∃x. (A(x) ∧ B(x))

"No As are Bs" "Some As aren't Bs" $\forall x. (A(x) \rightarrow \neg B(x)) \qquad \exists x. (A(x) \land \neg B(x))$

> It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- Person(p), which states that p is a person, and
- Loves(x, y), which states that x loves y,

write a sentence in first-order logic that means "every person loves someone else."

Answer at
https://cs103.stanford.edu/pollev

Every person loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

"All As are Bs"

 $\forall x. \ (A(x) \rightarrow B(x))$

∀p. (Person(p) →
p loves some other person

"All As are Bs" $\forall x. (A(x) \rightarrow B(x))$ $\forall p. (Person(p) \rightarrow p \text{ loves some other person})$

∀*p*. (*Person*(*p*) → *there is some other person that p loves*

 $\forall p. (Person(p) \rightarrow there is a person other than p that p loves$

 $\forall p. (Person(p) \rightarrow there is a person q, other than p, where p loves q$

∀p. (Person(p) → there is a person q, other than p, where p loves q

∀*p*. (*Person*(*p*) → *there is a person q, other than p, where p loves q*

"Some As are Bs"

 $\exists x. (A(x) \land B(x))$

$\forall p. (Person(p) \rightarrow \exists q. (Person(q) \land, other than p, where p loves q$

"Some As are Bs"

 $\exists x. (A(x) \land B(x))$

```
\forall p. (Person(p) \rightarrow \exists q. (Person(q) \land, other than p, where p loves q)
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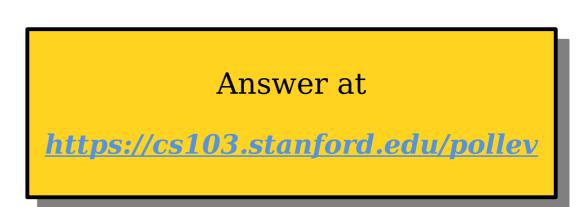
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 \forall p. (Person(p) \rightarrow \exists q. (Person(q) \land p \neq q \land p \text{ loves } q \land p \text{ loves } q \land p \text{ loves } q
```

```
 \forall p. (Person(p) \rightarrow \exists q. (Person(q) \land p \neq q \land Loves(p, q))
```

Using the predicates

- Person(p), which states that p is a person, and
- Loves(x, y), which states that x loves y,

write a sentence in first-order logic that means "there is a person that everyone else loves."



There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

"Some As are Bs"

 $\exists x. (A(x) \land B(x))$

∃p. (Person(p) ∧ everyone else loves p

"Some As are Bs"

 $\exists x. (A(x) \land B(x))$

∃p. (Person(p) ∧ everyone else loves p ∃p. (Person(p) ∧ every other person q loves p ∃p. (Person(p) ∧
 every person q, other than p, loves p

∃p. (Person(p) ∧
 every person q, other than p, loves p

"All As are Bs" $\forall x. (A(x) \rightarrow B(x))$

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\exists p. (Person(p) \land \forall q. (Person(q) \land p \neq q \rightarrow q loves p))
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"All As are Bs" $\forall x. (A(x) \rightarrow B(x))$

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\exists p. (Person(p) \land \forall q. (Person(q) \land p \neq q \rightarrow q loves p)
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```
\exists p. (Person(p) \land \forall q. (Person(q) \land p \neq q \rightarrow Loves(q, p))
```

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "Every person loves someone else"

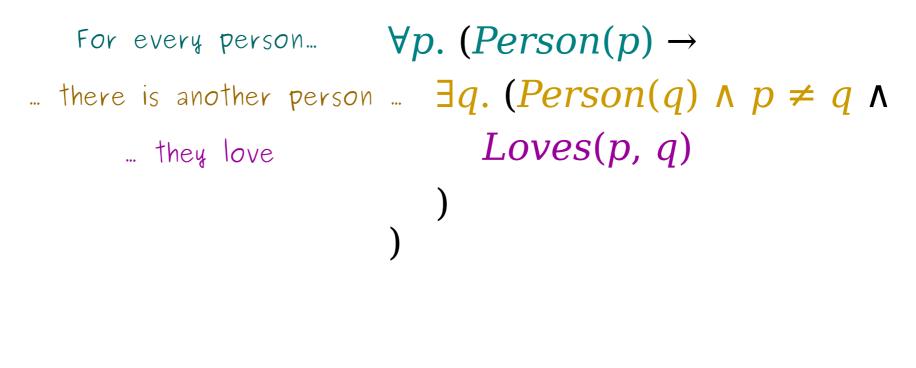
For every person... $\forall p. (Person(p)) \rightarrow$... there is another person ... $\exists q. (Person(q) \land p \neq q \land$... they love Loves(p, q)

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "There is someone everyone else loves."

There is a person... ... that everyone else ... $\exists p. (Person(p) \land$ $\forall q. (Person(q) \land p \neq q \rightarrow$ Loves(q, p))

For Comparison



There is a person...

... that everyone else ...

... loves.

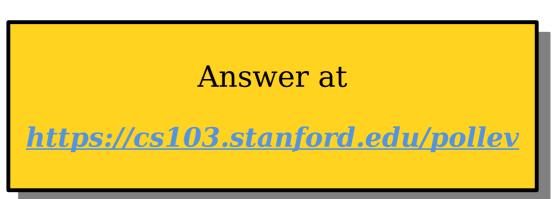
 $\exists p. (Person(p) \land \forall q. (Person(q) \land p \neq q \rightarrow Loves(q, p))$

• Consider these two first-order formulas:

 $\forall m. \exists n. m < n.$

 $\exists n. \forall m. m < n.$

- Pretend for the moment that our world consists purely of natural numbers, so the variables *m* and *n* refer specifically to natural numbers.
- One of these statements is true. The other is false.
- Which is which?
- Why?



• Consider these two first-order formulas:

 $\forall m. \exists n. m < n.$

 $\exists n. \forall m. m < n.$

• This says

for every natural number *m*, there's a larger natural number *n*.

- This is true: given any $m \in \mathbb{N}$, we can choose n to be m + 1.
- Notice that we can pick *n* based on *m*, and we don't have to pick the same *n* each time.

• Consider these two first-order formulas:

 $\forall m. \exists n. m < n.$

 $\exists n. \forall m. m < n.$

• This says

there is a natural number *n* that's larger than every natural number *m*

- This is false: no natural number is bigger than every natural number.
- Because $\exists n \text{ comes first}$, we have to make a single choice of n that works regardless of what we choose for m.

• The statement

∀*x*. ∃*y*. *P*(*x*, *y*)

means "for any choice of x, there's some choice of y where P(x, y) is true."

• The choice of *y* can be different every time and can depend on *x*.

• The statement

∃*x*. ∀*y*. *P*(*x*, *y*)

means "there is some x where for any choice of y, we get that P(x, y) is true."

• Since the inner part has to work for any choice of *y*, this places a lot of constraints on what *x* can be.

Order matters when mixing existential and universal quantifiers!

Time-Out for Announcements!

Problem Set Two

- Problem Set One was due today at 1:00PM.
 - You can extend the deadline to 1:00PM Saturday using one of your late days. As usual, no late submissions will be accepted beyond 1:00PM Saturday without prior approval.
 - We anticipate grades being released next Wednesday.
 - Regret Clause deadline will be Tuesday, 1 PM.
- Problem Set Two goes out today. It's due next Friday at 1:00PM.
 - Explore first-order logic!
 - Expand your proofwriting toolkit!
- We have some **online readings** for this problem set.
 - *Guide to Logic Translations*: more on converting from English to FOL.
 - *Guide to Negations*: information about how to negate formulas.
 - First-Order Translation Checklist: details on how to check your work.

Next week...

No classes on Monday. :)

Back to CS103!

Mechanics: Negating Statements

	When is this true?	When is this false?
$\forall x. P(x)$	For all objects <i>x</i> , <i>P</i> (<i>x</i>) is true.	There is an <i>x</i> where <i>P</i> (<i>x</i>) is false.
$\exists x. P(x)$	There is an x where P(x) is true.	For all objects <i>x</i> , <i>P</i> (<i>x</i>) is false.
$\forall x. \neg P(x)$	For all objects <i>x,</i> <i>P</i> (<i>x</i>) is false.	There is an x where P(x) is true.
$\exists x. \neg P(x)$	There is an <i>x</i> where <i>P</i> (<i>x</i>) is false.	For all objects <i>x,</i> <i>P</i> (<i>x</i>) is true.

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$\forall x. \neg P(x)$	For all objects <i>x,</i> <i>P</i> (<i>x</i>) is false.	There is an x where P(x) is true.
$\exists x. \neg P(x)$	There is an x where P(x) is false.	For all objects <i>x</i> , <i>P</i> (<i>x</i>) is true.

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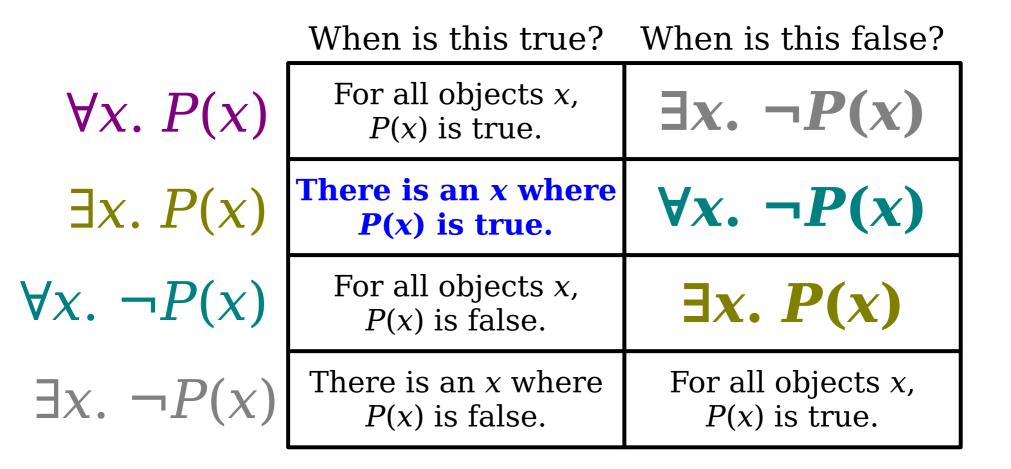
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$\forall x. \neg P(x)$	For all objects <i>x</i> , <i>P</i> (<i>x</i>) is false.	There is an x where P(x) is true.
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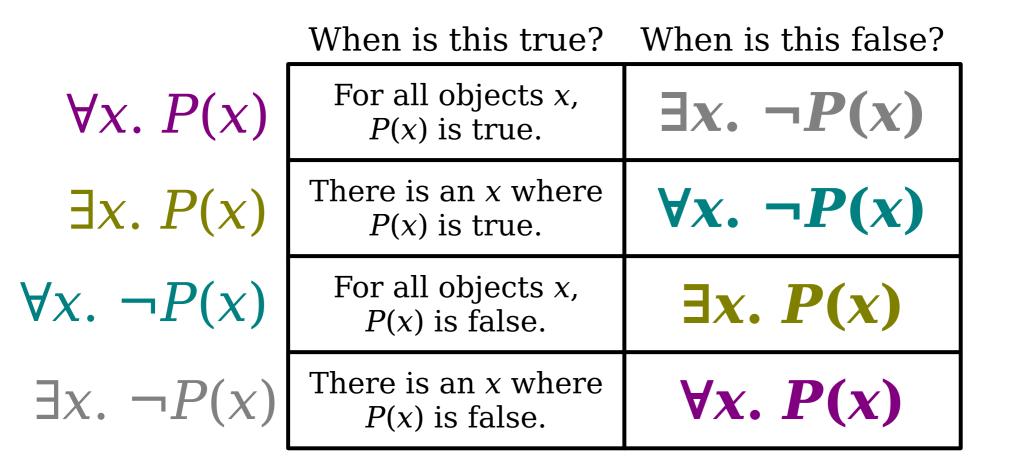
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$\exists x. \neg P(x)$	There is an <i>x</i> where <i>P</i> (<i>x</i>) is false.	$\forall x. P(x)$



Negating First-Order Statements

• Use the equivalences

 $\neg \forall x. A$ is equivalent to $\exists x. \neg A$ $\neg \exists x. A$ is equivalent to $\forall x. \neg A$ to negate quantifiers.

- Mechanically:
 - Push the negation across the quantifier.
 - Change the quantifier from \forall to \exists or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

 $\forall x. \exists y. Loves(x, y)$ ("Everyone loves someone.")

$$\neg \forall x. \exists y. Loves(x, y) \\ \exists x. \neg \exists y. Loves(x, y) \\ \exists x. \forall y. \neg Loves(x, y) \end{cases}$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

• The following equivalences are useful when negating statements in first-order logic:

 $\neg(p \land q)$ is equivalent to $p \rightarrow \neg q$ $\neg(p \rightarrow q)$ is equivalent to $p \land \neg q$

- These identities are useful when negating statements involving quantifiers.
 - $\boldsymbol{\Lambda}$ is used in existentially-quantified statements.
 - \rightarrow is used in universally-quantified statements.
- When pushing negations across quantifiers, we strongly recommend using the above equivalences to keep \rightarrow with \forall and \land with \exists .

Negating Quantifiers

• What is the negation of the following statement, which says "there is a cute puppy"?

 $\exists x. (Puppy(x) \land Cute(x))$



https://cs103.stanford.edu/pollev

Negating Quantifiers

• What is the negation of the following statement, which says "there is a cute puppy"?

$\exists x. (Puppy(x) \land Cute(x))$

• We can obtain it as follows:

 $\neg \exists x. (Puppy(x) \land Cute(x)) \\ \forall x. \neg (Puppy(x) \land Cute(x))$

 $\forall x. (Puppy(x) \rightarrow \neg Cute(x))$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

 $\exists S. (Set(S) \land \forall x. x \notin S) \\ ("There is a set with no elements.")$

 $\neg \exists S. (Set(S) \land \forall x. x \notin S) \\ \forall S. \neg (Set(S) \land \forall x. \neg x \notin S) \\ \forall S. (Set(S) \rightarrow \neg \forall x. x \notin S) \\ \forall S. (Set(S) \rightarrow \exists x. \neg (x \notin S)) \\ \forall S. (Set(S) \rightarrow \exists x. x \in S) \end{cases}$

("Every set contains at least one element.")

Restricted Quantifiers

Quantifying Over Sets

• The notation

 $\forall x \in S. P(x)$

means "for any element x of set S, P(x)holds." (It's vacuously true if S is empty.)

• The notation

$\exists x \in S. P(x)$

means "there is an element x of set S where P(x) holds." (It's false if S is empty.)

Quantifying Over Sets

• The syntax

 $\forall x \in S. P(x)$ $\exists x \in S. P(x)$

is allowed for quantifying over sets.

- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

 $\forall x \text{ with } P(x). \ Q(x)$ $\forall y \text{ such that } P(y) \land Q(y). \ R(y).$ $\exists P(x). \ Q(x)$ Expressing Uniqueness

Using the predicate

- *WayToFindOut(w)*, which states that *w* is a way to find out,

write a sentence in first-order logic that means "there is only one way to find out."

There is only one way to find out.

Something is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing besides w is a way to find out

∃w. (WayToFindOut(w) ∧ nothing besides w is way to find out

∃w. (WayToFindOut(w) ∧ anything that isn't w isn't a way to find out

∃w. (WayToFindOut(w) ∧ any thing x that isn't w isn't a way to find out

∃w. (WayToFindOut(w) ∧ $\forall x. (x \neq w \rightarrow x \text{ isn't a way to find out})$)

$\exists w. (WayToFindOut(w) \land \forall x. (x ≠ w → ¬WayToFindOut(x))$

$\exists w. (WayToFindOut(w) \land \forall x. (x ≠ w → ¬WayToFindOut(x)))$

```
\exists w. (WayToFindOut(w) \land \forall x. (WayToFindOut(x) → x = w)
```

```
\exists w. (WayToFindOut(w) \land \forall x. (WayToFindOut(x) → x = w)
```

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
 - there exists at least one object with that property, and that
 - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

∃!*x*. *P*(*x*)

 For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Next Time

• Functions

- How do we model transformations and pairings?
- First-Order Definitions
 - Where does first-order logic come into all of this?
- **Proofs with Definitions**
 - How does first-order logic interact with proofs?